STAT 821 HOMEWORK 4 SOLUTION

Question 1

Proof: $X_1, \dots, X_n \overset{iid}{\sim} U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$.

Let $Y_i = X_i - \theta + \frac{1}{2}$, $i = 1, \dots, n$, then $Y_1, \dots, Y_n \overset{iid}{\sim} U(0,1)$ and $Y_{(1)} = X_{(1)} - \theta + \frac{1}{2}$, $Y_{(n)} = X_{(n)} - \theta + \frac{1}{2}$.

It's known that $Y_{(1)} \sim Beta(1,n)$ and $Y_{(n)} \sim Beta(n,1)$. Thus

$$E(Y_{(1)}) = \frac{1}{n+1} \quad E(Y_{(n)}) = \frac{n}{n+1}$$

and

$$E(X_{(n)} - X_{(1)}) = E(Y_{(n)} - Y_{(1)}) = \frac{n-1}{n+1}$$

$$\text{i.e.} \quad E\left(X_{(n)} - X_{(1)} - \frac{n-1}{n+1}\right) = 0$$

Let

$$h(I) = X_{(n)} - X_{(1)} - \frac{n-1}{n+1}$$

, then the distribution of $X_{(n)} - X_{(1)}$ does not depend on θ.

$$E(h(I)) = 0 \quad \forall\, \theta$$

However, $P(h(I) = 0) \neq 1$, so $T(X) = (X_{(1)}, X_{(n)})$ is not complete.

Question 2

$f(x,y)$ has continuous partial derivatives of the first and second order on R^2.

$$\nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad H(f(x,y)) = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}$$

$\det\big(H(f(x,y))\big) = 4 > 0$ and the matrix is diagonal with positive diagonal elements. Thus $H(f(x,y))$ is positive definite and $f(x,y)$ is convex. At point $(1,0)$, the support hyperplane $L(X)$ is

$$\begin{aligned} L(X) &= f(1,0) + \nabla f(1,0)\ (<x,y> - <1,0>) \\ &= 1 + <2,0><x-1,y> \\ &= 2x - 1 \end{aligned}$$

Question 3

Proof: First show $-l(w)$ is convex.

$$-l(w) = \sum_{i=1}^{n} \log[1 + \exp(-y_i w^T x_i)]$$

$$\nabla(-l(w)) = \sum_{i=1}^{n} \begin{bmatrix} \frac{(-y_i x_{i1}) \exp(-y_i w^T x_i)}{1 + \exp(-y_i w^T x_i)} \\ \vdots \\ \frac{(-y_i x_{ik}) \exp(-y_i w^T x_i)}{1 + \exp(-y_i w^T x_i)} \end{bmatrix}$$

$$H(-l(w)) = \sum_{i=1}^{n} \begin{bmatrix} \frac{(-y_i x_{i1})^2 \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} & \cdots & \frac{(-y_i)^2 x_{i1} x_{ik} \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} \\ \vdots & & \vdots \\ \frac{(-y_i)^2 x_{i1} x_{ik} \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} & \cdots & \frac{(-y_i x_{ik})^2 \exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} \end{bmatrix}$$

$$= \sum_{i=1}^{n} \frac{\exp(-y_i w^T x_i)}{[1 + \exp(-y_i w^T x_i)]^2} x_i x_i^T$$

$H(-l(w))$ is positive definite since $x_i x_i^T$ is positive definite and hence $-l(w)$ is strictly convex.

Let

$$f(w_0) = -l(w_0) = -m$$

be the minimum and w_0 denote MLE. Suppose $\exists\ w_1$ s.t.

$$f(w_1) = f(w_0) = -m \quad \text{and} \quad w_1 \neq w_0$$

Then $\forall\ 0 \leq r \leq 1$,

$$rf(w_0) + (1-r)f(w_1) > f(rw_0 + (1-r)w_1)$$

by convexity of $-l(w)$. In other words,

$$-m > f(rw_0 + (1-r)w_1)$$

This is an contradiction to the fact that $-m$ is the minimum of $f(w)$. Thus MLE is unique.

Question 4

Proof:

$$\begin{aligned} E_F[h(x)] &= \int h(x)\,dF \\ &= \int \int_0^{h(x)} dt\,dF \\ &= \int_0^{h(x)} \int_{\{x\in R^k: h(x)>t\}} dF\,dt \\ &= \int_0^{\infty} F(h(x)>t)\,dt \end{aligned}$$

Similarly

$$E_G[h(x)] = \int_0^{\infty} G(h(x)>t)\,dt$$

We have

$$\begin{aligned} & E_G[h(x)] - E_F[h(x)] \\ =& \int_0^{\infty} [G(h(x)>t) - F(h(x)>t)]\,dt \\ =& \int_0^{\infty} [1-G(h(x)<t)] - [1-F(h(x)<t)]\,dt \\ =& \int_0^{\infty} [F(h(x)<t) - G(h(x)<t)]\,dt > 0 \qquad (*) \end{aligned}$$

Notice that the set $H = \{x : h(x) \le t\}$ is a convex set. This is because $\forall\, x, y \in H$

$$h(rx-(1-r)y) \le rh(x) + (1-r)h(y) \le t$$

So $rx + (1-r)y \in H$.

Thus, (*) implies there is a convex set $A \in R^k$, with $\underset{\sim}{0} \in A$ s.t. $\forall\, t_0 \in A$,

$$F(h(x) \le t_0) - G(h(x) \le t_0) \ge 0$$

$$\Rightarrow\ F(A) \ge G(A) \quad \text{for such A}$$

Question 5

(a)

$$
\begin{aligned}
\nabla f(x,y) &= \begin{pmatrix} -\alpha x^{\alpha-1}y^{1-\alpha} \\ -x^{\alpha}(1-\alpha)y^{-\alpha} \end{pmatrix} \\
H(f(x,y)) &= \begin{pmatrix} -\alpha(\alpha-1)x^{\alpha-2}y^{1-\alpha} & -\alpha(1-\alpha)x^{\alpha-1}y^{-\alpha} \\ -\alpha(1-\alpha)x^{\alpha-1}y^{-\alpha} & \alpha(1-\alpha)x^{\alpha}y{-}\alpha-1 \end{pmatrix}
\end{aligned}
$$

$$
\det(H) = \alpha^2(1-\alpha)^2x^{2\alpha-2}y^{-2\alpha} - \alpha^2(1-\alpha)^2x^{2\alpha-2}y^{-2\alpha} = 0
$$

$$
tr(H) = \alpha(1-\alpha)x^{\alpha-2}y^{-\alpha-1}[x^2+y^2] > 0
$$

The eigenvalue $\lambda's$ are the roots of the equation

$$
\lambda^2 - \lambda tr(H) + \det(H) = 0
$$

$$
\Rightarrow \; \lambda(\lambda - tr(H)) = 0
$$

$$
\Rightarrow \; \lambda = 0 \quad \text{or} \quad \lambda = tr(H) > 0
$$

Both of the roots are non-negative and hence the Hessian matrix is positive semidefinite.

$$
\Rightarrow \; f(x,y) = -x^{\alpha}y^{1-\alpha} \qquad \forall\, 0 < \alpha < 1
$$

is convex on $\{(x,y) \in R^2 : x > 0, y > 0\}$.

(b) Use Jensen's Inequality

$$
E(f(x)) \geq f(E(x)))
$$

for convex $f(\cdot)$. Here we have

$$
f(x,y) = -x^{\alpha}y^{1-\alpha} \qquad \forall\, 0 < \alpha < 1
$$

So

$$
E[-x^{\alpha}y^{1-\alpha}] \geq -(EX)^{\alpha}(EY)^{1-\alpha}
$$

i.e.

$$
E(X^{\alpha}Y^{1-\alpha}) \leq (EX)^{\alpha}(EY)^{1-\alpha}
$$

(c) We first show that the natural parameter space

$$\Theta = \{\eta : \int \exp(\eta' T) d\mu < \infty\}$$

is convex.

Suppose $\eta_1, \eta_2 \in \Theta$ and for $0 \le r \le 1$

$$\begin{aligned}
& \int \exp(r\eta_1' T + (1-r)\eta_2' T)\, d\mu \\
\propto\ & E\big[(e^{\eta_1' T})^r (e^{\eta_2' T})^{1-r}\big] \\
\le\ & \big[E(e^{\eta_1' T})^r (E e^{\eta_2' T})^{1-r}\big] \\
\propto\ & \left(\int \exp(\eta_1' T)\, d\mu\right)^r \left(\int \exp(\eta_2' T)\, d\mu\right)^{1-r} \\
<\ & \infty \qquad \text{since } \eta_1, \eta_2 \in \Theta
\end{aligned}$$

Therefore $r\eta_1 + (1-r)\eta_2 \in \Theta$ and hence Θ is convex.

$A(\eta)$ is defined on a convex set Θ since $Cov(T) = A(\eta)$ and $Cov(T)$ is positive semidefinite. Thus $A(\eta)$ is a convex function on Θ.